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KPB (B)

Tugas Integral Lipat

Jumat, 3 November 2023

10\. Diberikan integral : $\displaystyle\int_0^{\sqrt{2}}\int_y^{\sqrt{4-y^2}} \frac{1}{1+x^2+y^2}\,dx\,dy$

a). Sketsalah daerah integrasinya !

Batas-Batas :

-> Terhadap Sumbu-x (horizontal)

- $x = \sqrt{4-y^2}$

$x^2 = 4-y^2$

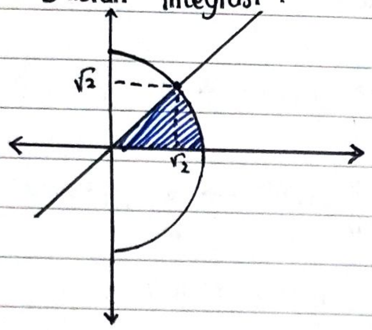
$x^2+y^2 = 2^2 \rightarrow$ lingkaran pusat $(0,0)$ dan $r=2$

- $x = y$

-> Terhadap Sumbu-y (Vertikal)

- $y = \sqrt{2}$
- $y = 0$

▶ Sketsa Daerah Integrasi :



b). Dapatkan nilai dari integral tersebut !

misal : $x = r\cos\theta$ $y = r\sin\theta$

maka $\rightarrow$ $x^2+y^2 = (r\cos\theta)^2 + (r\sin\theta)^2$

$= r^2\left[\cos^2\theta + \sin^2\theta\right]$

$= r^2$

* Jacobian :

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$= r\cos^2\theta - (-r\sin^2\theta)$

$= r(\cos^2\theta + \sin^2\theta)$

$= r$

* Batasannya menjadi :

-> $r = 0$, hingga $r = 2$

-> $\theta = 0$ — $\theta = \frac{\pi}{4}$

$$\int_0^{\sqrt{2}}\int_y^{\sqrt{4-y^2}} \frac{1}{1+x^2+y^2}\,dx\,dy = \iint_D G(r,\theta)\,|J|\,dr\,d\theta$$

$$\Rightarrow \int_0^{\pi/4}\int_0^{2} \frac{1}{1+r^2}\cdot r\;dr\,d\theta$$

misal : $1+r^2 = p \longrightarrow dp = 2r\,dr$

$r\,dr = \frac{dp}{2}$

BA $\rightarrow p = 5$

BB $\rightarrow p = 1$

$$\int_0^{\pi/4}\int_1^{5} \frac{1}{p}\frac{dp}{2}\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4}\left[\left[\ln p\right]_1^5\right]d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4}\left[\ln 5\right]d\theta$$

$$= \frac{1}{2}\ln 5\left(\frac{\pi}{4}\right)$$

$$= \frac{\pi}{8}\ln 5$$

Jawab : $\frac{\pi}{8}\ln 5$

16). Gunakan koordinat kutub untuk menyelesaikan:

a). $\iint_D (x^2+y^2)\,dx\,dy$, dimana D adalah yang dibatasi lingkaran $x^2+y^2 = 2ax$

* Bentuk umum persamaan lingkaran dengan pusat (a,b) dan jari-jari r :

$(x-a)^2 + (y-b)^2 = r^2$

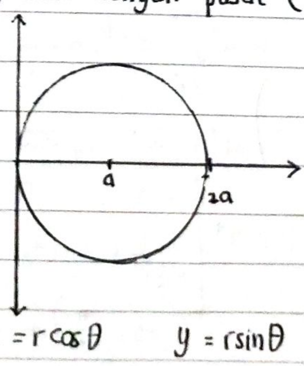
maka, persamaan lingkaran :

$x^2 - 2ax + y^2 = 0$

$x^2 - 2ax + a^2 - a^2 + y^2 = 0$

$(x-a)^2 + y^2 = a^2$

$\Rightarrow$ Lingkaran dengan pusat $(a,0)$ dan jari-jari a.



misal : $x = r\cos\theta$ $y = r\sin\theta$

$x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta = 2ar\cos\theta$

$r^2(\cos^2\theta + \sin^2\theta) = 2ar\cos\theta$

$r^2 = 2ar\cos\theta$

$r = 2a\cos\theta$

Jacobian :

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta - (-r\sin^2\theta)$$

$$= r(\cos^2\theta + \sin^2\theta)$$

$$= r$$

$$\iint_D (x^2+y^2)\,dx\,dy = \int_{-\pi/2}^{\pi/2}\int_0^{2a\cos\theta} r^2 \cdot |J|\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\int_0^{2a\cos\theta} r^2 \cdot r\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\left[\int_0^{2a\cos\theta} r^3\,dr\right]d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\left[\frac{1}{4}r^4\right]_0^{2a\cos\theta}d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \frac{1}{\cancel{4}}\cdot \overset{4}{\cancel{16}}a^4\cos^4\theta\,d\theta$$

$$= 4a^4\int_{-\pi/2}^{\pi/2}(\cos^2\theta)^2\,d\theta$$

※ $\cos 2\theta = 2\cos^2\theta - 1$

$\cos^2\theta = \frac{\cos 2\theta + 1}{2}$

$$= 4a^4\int_{-\pi/2}^{\pi/2}\left(\frac{\cos 2\theta + 1}{2}\right)^2 d\theta$$

$$= a^4\int_{-\pi/2}^{\pi/2}\left(\cos^2 2\theta + 2\cos 2\theta + 1\right)d\theta$$

$$= a^4\int_{-\pi/2}^{\pi/2}\left(\frac{\cos 4\theta}{2} + \frac{1}{2} + 2\cos 2\theta + 1\right)d\theta$$

$$= a^4\int_{-\pi/2}^{\pi/2}\frac{\cos 4\theta}{2} + 2\cos 2\theta + \frac{3}{2}\,d\theta$$

$$= a^4\left[\frac{\sin 4\theta}{8} + \frac{2\sin 2\theta}{2} + \frac{3}{2}\theta\right]_{-\pi/2}^{\pi/2}$$

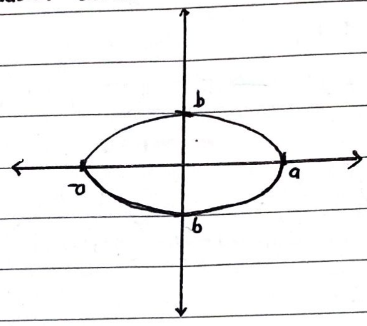
$$= a^4\left[\left(\frac{\sin 2\pi}{8} + \sin(\pi) + \frac{3}{2}\cdot\frac{\pi}{2}\right) - \left(\frac{\sin(-2\pi)}{8} + \sin(-\pi) + \left(-\frac{\pi}{2}\cdot\frac{3}{2}\right)\right)\right]$$

$$= a^4\left[\frac{3\pi}{4} + \frac{3\pi}{4}\right]$$

$$= \frac{6\pi a^4}{4} = \frac{3}{2}\pi a^4$$

17). Selesaikan $\iint_D \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\,dx\,dy$ dengan Substitusi $\frac{x}{a} = r\cos\theta$ dan $\frac{y}{b} = r\sin\theta$ di mana D adalah daerah yang dibatasi $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Luasan awal :



※ Transformasi dengan mensubstitusi $\frac{x}{a} = r\cos\theta$ dan $\frac{y}{b} = r\sin\theta$.

misal : • $\frac{x}{a} = r\cos\theta \rightarrow \frac{x^2}{a^2} = r^2\cos^2\theta$

• $\frac{y}{b} = r\sin\theta \rightarrow \frac{y^2}{b^2} = r^2\sin^2\theta$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = r^2\cos^2\theta + r^2\sin^2\theta$$

$$= r^2(\cos^2\theta + \sin^2\theta)$$

$$= r^2$$

Karena $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, maka $r = 1$

※ Jacobian

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix}$$

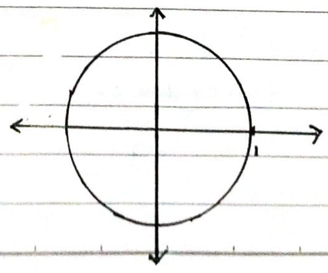
$$= \begin{vmatrix} a\cos\theta & -ar\sin\theta \\ b\sin\theta & br\cos\theta \end{vmatrix}$$

$$= abr\cos^2\theta - (-abr\sin^2\theta)$$

$$= abr\left[\cos^2\theta + \sin^2\theta\right]$$

$$= abr$$

※ Luasan baru :



$$\iint_D \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dx\,dy = \int_0^{2\pi}\int_0^1 \sqrt{1-r^2}\cdot abr\,dr\,d\theta$$

↳ $ab\int_0^{2\pi}\left[\int_0^1 r\sqrt{1-r^2}\,dr\right]d\theta$

Misal : $p = 1-r^2$ BA : $p = 1-1^2 = 0$

$dp = -2r\,dr$

$r\,dr = -\frac{dp}{2}$ BB : $p = 1$

$$= ab\int_0^{2\pi}\int_1^0 \sqrt{p}\,\frac{dr}{-2}\,d\theta$$

$$= -\frac{ab}{2}\int_0^{2\pi}\left[\frac{2}{3}r\sqrt{r}\right]_1^0 d\theta$$

$$= -\frac{ab\cancel{2}}{\cancel{2}3}\int_0^{2\pi}(-1)\,d\theta$$

$$= \frac{ab}{3}\cdot\theta\Big]_0^{2\pi}$$

Jawab : $\frac{2\pi ab}{3}$

$$= \frac{2\pi ab}{3}$$

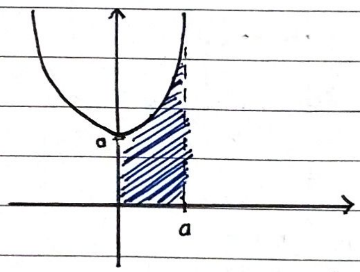
16 b). $\int_0^a\int_0^{\sqrt{a^2+x^2}} \sqrt{x^2+y^2}\,dx\,dy$

Gambar Daerah :

Batasan

-> Searah sumbu y : · $y = \sqrt{a^2+x^2}$
· $y = 0$

-> Searah sumbu x : · $x = a$
· $x = 0$



Misal : $x = r\cos\theta$ dan $y = r\sin\theta$

$x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta$

$= r^2\left[\cos^2\theta+\sin^2\theta\right]$

$= r^2$

Jacobian

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

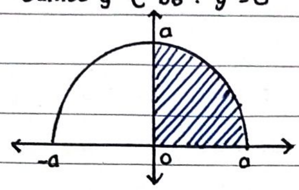
$$= r\cos^2\theta - (-r\sin^2\theta)$$

$$= r(\cos^2\theta+\sin^2\theta)$$

$$= r$$

Karena daerahnya susah ketika ditransformasikan, ubah batas menjadi $y = \sqrt{a^2-x^2}$

•> Batasan : Sumbu y $\begin{cases} BA : y = \sqrt{a^2-x^2} \\ BB : y = 0 \end{cases}$ | Sumbu x $\begin{cases} BA : x = a \\ BB : x = 0 \end{cases}$



Misalkan : $x = r\cos\theta$; $y = r\sin\theta$

$x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta$

$= r^2(\cos^2\theta+\sin^2\theta)$

$= r^2$

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta - (-r\sin^2\theta)$$

$$= r(\cos^2\theta+\sin^2\theta)$$

$$= r$$

$$\iint_D \sqrt{x^2+y^2}\,dx\,dy = \iint_D \sqrt{r^2}\cdot|J|\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^a \sqrt{r^2}\cdot r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^a r^2\,dr\,d\theta$$

$$= \int_0^{\pi/2} \frac{1}{3}r^3\Big]_0^a\,d\theta$$

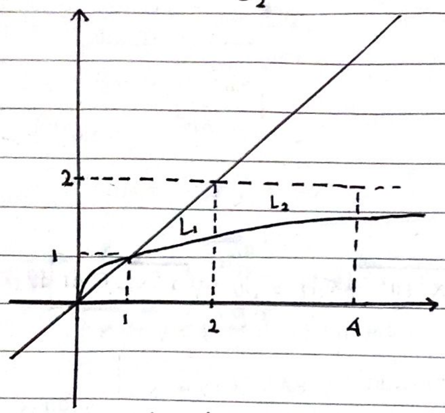
$$= \int_0^{\pi/2} \frac{1}{3}a^3\,d\theta$$

$$= \frac{1}{3}a^3\,\theta\Big]_0^{\pi/2}$$

$$= \frac{1}{3}a^3\cdot\frac{\pi}{2}$$

$$= \frac{\pi}{6}a^3$$

14. $\underbrace{\int_1^2\int_{\sqrt{x}}^{x} \sin\frac{\pi x}{2y}\,dy\,dx}_{L_1} + \underbrace{\int_2^4\int_{\sqrt{x}}^{2} \sin\frac{\pi x}{2y}\,dy\,dx}_{L_2} = \frac{4(\pi+2)}{\pi^3}$



Ubah urutan pengintegralan untuk langsung menentukan luasan total.

▶ Searah x : · Kanan : $x = y^2$

· Kiri : $x = y$

▶ Searah y : · Atas : $y = 2$

· Bawah : $y = 1$

※ Hitung Integrasi :

$$\int_1^2\left[\int_y^{y^2}\sin\frac{\pi}{2y}x\,dx\right]dy = \int_1^2\left[\frac{1}{\pi}\cdot 2y\left(-\cos\frac{\pi x}{2y}\right)\right]_y^{y^2}dy$$

$$= \frac{-2}{\pi}\int_1^2\left[y\left(\cos\frac{\pi}{2}y - \cos\frac{\pi}{2}\right)\right]dy$$

$$= -\frac{2}{\pi}\int_1^2 y.\cos\frac{\pi}{2}y\,dy$$

※ misal : $u = y$ $\quad dv = \cos\frac{\pi}{2}y\,dy$

$du = dy$ $\quad v = \frac{2}{\pi}\sin\frac{\pi}{2}y$

$$\int_1^2 y\cos\frac{\pi}{2}y\,dy = \frac{2y}{\pi}\sin\frac{\pi}{2}y\Big]_1^2 - \int_1^2\frac{2}{\pi}\sin\frac{\pi}{2}y\,dy$$

$$= \left(\frac{4}{\pi}\sin\pi - \frac{2}{\pi}\sin\frac{\pi}{2}\right) - \frac{2}{\pi}\left[\frac{2}{\pi}\left(-\cos\frac{\pi}{2}y\right)\right]_1^2$$

$$= -\frac{2}{\pi} - \frac{4}{\pi^2}\left(-(-1)+0\right)$$

$$= -\frac{2}{\pi} - \frac{4}{\pi^2}$$

$$\int_1^2\int_y^{y^2}\sin\frac{\pi}{2y}x\,dx\,dy = -\frac{2}{\pi}\left[-\frac{2}{\pi}-\frac{4}{\pi^2}\right]$$

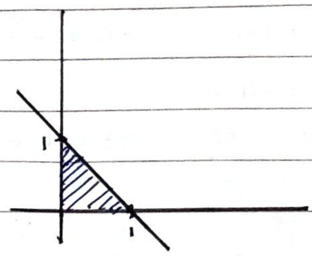
$$= \frac{4}{\pi^2}+\frac{8}{\pi^3} = \frac{4(\pi+2)}{\pi^3} \;//$$

TERBUKTI //

18). $\iint_D \cos\left(\frac{x-y}{x+y}\right)dx\,dy = \frac{\sin 1}{2}$

D daerah yang dibatasi $x+y=1$, $x=0$, dan $y=0$

Daerah :



Karena bentuk tersebut susah diintegralkan, maka misalkan saja :

- $x - y = u$
- $x + y = v$

$$\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}u\\ v\end{bmatrix}$$

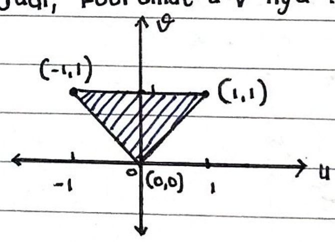
Pada gambar, daerah yang diarsir dibatasi oleh titik (0,1), (1,0), dan (0,0). Maka, titiknya pada koordinat u-v adalah.

$\to \begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix} = \begin{bmatrix}-1\\ 1\end{bmatrix}$

$\to \begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} = \begin{bmatrix}1\\ 1\end{bmatrix}$

$\to \begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}0\\ 0\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$

▶ Jadi, koordinat u-v nya :



Persamaan Garisnya :

$\to$ melewati (1,1) dan (0,0)

$$\frac{u_2-u_1}{v_2-v_1} = \frac{u-u_1}{v-v_1}$$

$$\frac{1}{1} = \frac{u}{v}$$

$$u = v$$

$\cdot\!>$ Melewati $(0,0)$ dan $(-1,1)$

$$\frac{u_2-u_1}{v_2-v_1} = \frac{u-u_1}{v-v_1}$$

$$\frac{-1}{1} = \frac{u}{v}$$

$$u = -v$$

\# Jacobian

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

$$= \frac{1}{\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}}$$

$$= \frac{1}{\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}}$$

$$= \frac{1}{1-(-1)}$$

$$= \frac{1}{2}$$

$$\iint_D \cos\left(\frac{x-y}{x+y}\right) dx\, dy$$

$$= \int_0^1 \int_{-v}^{v} \cos \frac{u}{v} \cdot \frac{1}{2}\, du\, dv$$

$$= \frac{1}{2} \int_0^1 v \sin \frac{u}{v} \Big]_{-v}^{v} dv$$

$$= \frac{1}{2} \int_0^1 v(\sin 1 - \sin(-1))\, dv$$

$$= \frac{1}{2} \int_0^1 v(\sin 1 + \sin 1)\, dv$$

$$= \frac{1}{2}\, 2.\sin 1 \int_0^1 v\, dv$$

$$= \sin 1 \left[\tfrac{1}{2} v^2 \right]_0^1$$

$$= \sin 1 . \left(\tfrac{1}{2}\right)$$

$$= \frac{\sin 1}{2} //$$

20). Hitung $\iint_D e^{-(x^2+y^2)}\, dx\, dy$.

D = daerah bidang datar seluruh kuadran pertama.

misal : $x = r\cos\theta$ $\quad y = r\sin\theta$

$x^2 = r^2\cos^2\theta \quad y^2 = r^2\sin^2\theta$

$$x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta$$

$$= r^2\left[\cos^2\theta + \sin^2\theta\right]$$

$$= r^2(1)$$

$$= r^2$$

\# Jacobian

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta - (-r\sin^2\theta)$$

$$= r\left[\cos^2\theta + \sin^2\theta\right]$$

$$= r(1)$$

$$= r$$

Batas : $0 \le \theta \le \frac{\pi}{2}$ dan $0 \le r \le \infty$

$$\int_0^{\pi/2}\int_0^{\infty} e^{-r^2}.\, |J|\, dr\, d\theta$$

$$= \int_0^{\pi/2} \lim_{p\to\infty} \int_0^p e^{-r^2}.r\, dr\, d\theta$$

= misal : $k = -r^2$ $\quad$ BA : $k = -\infty$

$dk = -2r\, dr$ $\quad$ BB : $k = 0$

$$= \int_0^{\pi/2} \lim_{p\to-\infty} \int_0^p e^k \frac{dk}{-2}\, d\theta$$

$$= -\frac{1}{2}\int_0^{\pi/2}\left[\lim_{p\to-\infty}\int_0^p e^k\, dk.\right] d\theta$$

$$= -\frac{1}{2}\int_0^{\pi/2}\left[\lim_{p\to-\infty} .e^p - e^0\right] d\theta$$

$$= -\frac{1}{2}\int_0^{\pi/2}(e^{-\infty} - 1)\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} d\theta$$

$$= \frac{1}{2} . \frac{\pi}{2}$$

$$= \frac{\pi}{4} //$$

Jawab $= \frac{\pi}{4}$ //